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# ON DIVISOR THEORY IN AN ARCHIMEDIAN LATTICE-ORDERED SEMIGROUP

Dedicated to Emeritus Professor Mchio Nagumo on his 70th birthday

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The main purpose of this note is to consider a divisor theory of lattice-ordered semigroups (abbr. l-semigroups), and to show that an l-semigroup  $S$  is Artinian if and only if the cone of  $S$  has the divisor theory.

1. Introduction. Let  $L$  be an l-semigroup (not necessarily commutative), and let  $\Sigma$  be any multiplicatively closed subset of  $L$  such that for each element  $a \in L$  there is an element  $x \in \Sigma$  with  $x \leq a$ . Let  $\Delta$  be a commutative l-semigroup with unity quantity  $\xi$  such that (1)  $\xi$  is the greatest element of  $\Delta$ , (2)  $\Delta$  contains primes and (3) each element of  $\Delta$  is uniquely decomposed into primes apart from its commutativity.

An l-semigroup epimorphism  $f: a \mapsto f(a)$  from  $L$  to  $\Delta$  is called a right divisor theory of  $L$  if it satisfies the following conditions:

(1°) If for  $x, y \in \Sigma$ ,  $f(x)$  is divisible by  $f(y)$  in  $\Delta$ , then  $x$  is divisible by  $y$  on the right-hand side in  $L$ , i.e. if there is an element  $\gamma \in \Delta$  such that  $f(x) = \gamma f(y)$ , then there is an element  $c \in L$  such that  $x = cy$ .

(2°)  $\Sigma(\alpha) = \Sigma(\beta)$  implies  $\alpha = \beta$ , where  $\Sigma(\alpha)$  is the set of the elements of  $x \in \Sigma$  such that  $f(x)$  is divisible by  $\alpha \in \Sigma$ .

A left divisor theory is defined analogously.

A main purpose of this note is to prove the following

THEOREM. Let  $S$  be a conditionally complete lattice-ordered semigroup (abbr. cl-semigroup) with unity quantity  $e$ . Assume that the cone  $C = \{a \in S; a \leq e\}$  satisfies the ascending chain condition in the sense of quasi-equality (cf. DEFINITION 3) and has a join-

generator system  $\Sigma$  such that (a)  $\Sigma$  is closed under multiplication (b) every element of  $\Sigma$  is invertible in  $S$  and (c) every element  $s \in S$  is written as  $s = ax^{-1} = y^{-1}b$  where  $a, b \in C$  and  $x, y \in \Sigma$ . Then the following conditions are equivalent:

- (1<sub>r</sub>)  $C$  has a right divisor theory.
- (1<sub>l</sub>)  $C$  has a left divisor theory.
- (2)  $C$  is archimedean.
- (3)  $S$  is Artinian.

Let  $G$  be the group generated by  $\Sigma$  in  $S$ . Then  $S$  is a quotient semigroup of  $C$  by  $G \wedge C$  in the sense of [2], where  $\wedge$  will denote the intersection. The cone  $C$  of  $S$  is said to be archimedean, if whenever  $z^n x \leq e$  for  $n = 1, 2, \dots$  ( $x \in \Sigma$ ,  $z \in G$ ) imply  $z \leq e$ . Since  $z^n x \leq e \iff z^n \leq x \iff xz^n \leq e$ , there needs no distinction of "right" and "left" for archimedeanesness. An Artinian 1-semigroup is considered in the next section.

2. Artinian 1-semigroups. Let  $S$  be a cl-semigroup whose cone  $C$  has a join-generator system  $\Sigma$  with the conditions (a), (b) and (c) in the theorem mentioned above.

LEMMA 1. The group  $G$  generated by  $\Sigma$  in  $S$  is a join-generator system of  $S$ .

Proof. The any element  $a \in S$  there is an element  $x \in \Sigma$  such that  $ax \in C$ . That  $ax = \sup N$  for a subset  $N$  of  $\Sigma$ . Hence we have  $a = (\sup N)x^{-1} = \sup(Nx^{-1})$  where  $Nx^{-1} = \{ux^{-1}; u \in N\}$ . This means that  $G$  is a join-generator system of  $S$ .

LEMMA 2. For any two elements  $a$  and  $b$  of  $S$ ,  $X(a, b) = \{u \in G; ub \leq a\}$  is non-void. The set  $F(a, b) = \{s \in S; sb \leq a\}$  has an upper bound, and  $\sup F(a, b) = \sup X(a, b)$ .

Proof. Take an element  $x \in G$  such that  $x \leq a$ , and take  $y \in \Sigma$  such that  $yb \leq e$ . Then putting  $u = xy$  we have  $ub \leq a$ . It is easy to see that  $av^{-1}$  is an upper bound of  $F(a, b)$  for any  $v \in G$  with  $v \leq b$ . Let  $s = cx^{-1}$  be any element of  $F(a, b)$  where  $c \in C$ ,  $x \in \Sigma$ ; and put  $J = \{z \in \Sigma; z \leq c\}$ . Then since  $zx^{-1}b \leq cx^{-1}b = sb \leq a$  and  $zx^{-1}$

$\in G$ , we have  $s = cx^{-1} = (\sup J)x^{-1} = \sup(Jx^{-1}) \leq \sup X(a,b)$ . Hence  $\sup F(a,b) \leq \sup X(a,b)$ . The converse inequality is evident.

DEFINITION 1.  $a/b = \sup F(a,b)$  is called a right residual of  $a$  by  $b$ .

LEMMA 3. If  $a \in S$  and  $u \in G$ , then  $a/u = au^{-1}$ . In particular  $e/u = u^{-1}$ .

Proof. There is a subset  $A$  of  $G$  such that  $a/u = \sup A$ . Then for any  $z \in A$  we have  $zu \leq a$ ,  $z \leq au^{-1}$ ,  $a/u \leq au^{-1}$ . The converse inequality is evident.

The residual has the following properties:

- (1)  $a/(bc) = (a/c)/b$ .
- (2)  $(\inf A)/b = \inf \{a/b; a \in A\}$ , if either  $\inf A$  or the right-hand side exists.
- (3)  $a/(\sup B) = \inf \{a/b; b \in B\}$ , if either  $\sup B$  or the right-hand side exists.

It is clear that  $U(a) = \{u \in G; a \leq u\}$  is non-void for any  $a \in S$ .

DEFINITION 2.  $a^* = \inf U(a)$  is called a closure of  $a$ .  $a$  is said to be closed if  $a^* = a$ .

The following properties are immediate:

- (4)  $a \leq a^*$ .
- (5)  $a \leq b$  implies  $a^* \leq b^*$ .

LEMMA 4. If  $a$  is closed, then  $a/b$  is closed for any  $b \in S$ .

Proof. Let  $b = \sup B$  for a subset  $B$  of  $G$ . Then since  $a = \inf U(a)$  we have  $a/b = \inf U(a)/\sup B = \inf \{u/v; u \in U(a), v \in B\} = \inf \{uv^{-1}\} \geq \inf U(a/b) \geq a/b$  (by (2), (3) and LEMMA 3). Hence we obtain  $a/b = \inf U(a/b)$  as desired.

We have the following properties:

- (6)  $a^* = e/(e/a)$ .
- (7)  $e/a = e/a^*$ .
- (8)  $a^{**} = a^*$ .
- (9)  $a^*b^* \leq (a^*b^*)^* = (ab)^*$ .
- (10)  $(\sup A)^* = \sup(A^*)^*$ , if either  $\sup A$  or  $\sup A^*$  exists, where  $A^* = \{a^*; a \in A\}$ . In particular  $(a \cup b)^* = (a^* \cup b^*)^*$ .

(11)  $(\inf A^*)^* = \inf A^*$ , if  $\inf A^*$  exists. In particular  $(a^* \cap b^*)^* = a^* \cap b^*$ .

We define an operation " $\circ$ " by  $a^* \circ b^* = (ab)^*$ .

(12)  $(\sup A)^* \circ b^* = (\sup(A^* \circ b^*))^*$ ,  $b^* \circ (\sup A)^* = (\sup(b^* \circ A^*))^*$ , if  $\sup A$  exists.

Proof. Ad (6): For any  $u \in U(a)$  we have  $e/a \geq e/u = u^{-1}$ ,  $e/(e/a) \leq e/u^{-1} = u$ . Hence  $e/(e/a) \leq \inf U(a) = a^*$ . Conversely since  $a = \sup A$  for a suitable subset  $A$  of  $G$ , we have  $x^{-1} = e/x \geq e/a$  for any  $x \in A$ . Hence  $x = e/x^{-1} = e/(e/x) \leq e/(e/a)$  and hence  $a = \sup A \leq e/(e/a)$ . Thus we obtain  $a^* \leq e/(e/a)$  by LEMMA 4 and (5).

Ad (7): By (6) we have  $e/a^* = e/(e/(e/a)) = (e/a)^* \geq e/a$ . The converse inequality is evident. (8) is immediate by (6) and (7).

Ad (9): Since  $e/(ab)^* = e/(ab) = (e/b)/a = (e/b^*)/a = e/(ab^*)$ , we have  $(ab)^* = (ab)^{**} = e/(e/(ab)^*) = e/(e/(ab^*)) = (ab^*)^*$ . Now we can define left residuals and argue symmetrically as above. If  $u \in G$  then  $ua \leq e \iff a \leq u^{-1} \iff au \leq e$ . Hence we have  $e/a = a \setminus e$ , the left residual of  $e$  by  $a$ . This yields  $(ab)^* = (a^*b)^*$ , and the identity of (9) holds. (10), (11) and (12) are checked easily.

DEFINITION 3. Two elements  $a, b \in S$  are said to be quasi-equal, if  $a^* = b^*$ . In symbol:  $a \sim b$ .

(13)  $a \sim b$  implies  $e/a = e/b$ , and conversely.

(14)  $a^* \sim a$ .

(15)  $a^* \sim c$  implies  $a^* \geq c$ .

The above three are immediate. Put  $S^* = \{s^*; s \in S\}$ , and define  $a^* \vee b^* = (a^* \cup b^*)^* = (a \cup b)^*$ ,  $a^* \wedge b^* = (a^* \cap b^*)^* = a^* \cap b^*$  and  $a^* \wedge b^* = (a \cap b)^*$ . Then by using (8)  $\sim$  (12) we can show that  $(S^*, \circ, \vee, \wedge)$  is cl-semigroup, and similarly for  $(S^*, \circ, \vee, \wedge)$ .

DEFINITION 4. If the semigroup  $(S^*, \circ)$  is a group,  $S$  is called an Artinian 1-semigroup [3].

We can show that if  $S$  is Artinian,  $(S^*, \circ, \vee, \wedge)$  is an cl-group. Hence  $(S^*, \circ)$  is a commutative group, and  $(S^*, \vee, \wedge)$  is a distributive lattice. In this case  $e$  is maximally integral (cf. p. 12 in [1]). For it can be shown that  $C$  is archimedean if and only if the above

two meet operations coincide (cf. pp 13-14 in [1]).

3. Proof of THEOREM.  $(1_r) \Rightarrow (2)$ ; Let  $(C, \Delta, f)$  be a given divisor theory of the cone  $C$  of  $S$ , and let  $H$  be the restricted direct product of infinite cyclic groups, each of which is generated by a prime divisor in  $\Delta$ . Then it can be shown that  $f: C \rightarrow \Delta$  extends to a map  $f: S \rightarrow H$  by  $f: cz^{-1} \mapsto f(c)f(z)^{-1}$  where  $cz^{-1} \in S$ ,  $c \in C$ ,  $z \in \Sigma$ .  $f(cz^{-1})$  does not depend on the choice of the fractional representations.

Suppose that  $xu^n \leq e$ ,  $x \in \Sigma$ ,  $u \in G$  for  $n = 1, 2, \dots$ , and let  $f(x) = \pi_1^{\lambda_1} \dots \pi_r^{\lambda_r} \pi_{r+1}^{\lambda_{r+1}} \dots \pi_m^{\lambda_m}$  ( $\lambda_i > 0$ ),  $f(u) = \pi_1^{\mu_1} \dots \pi_r^{\mu_r} \pi_{r+1}^{\mu_{r+1}} \dots \pi_t^{\mu_t}$  ( $\mu_i > 0$  or  $< 0$ ) be the prime factorizations in  $H$ , where  $\pi_1, \dots, \pi_r$  are the common prime divisors. The since  $f(xc^n) \in \Delta$  we have  $\lambda_i + n\mu_i \geq 0$  ( $i = 1, \dots, r$ ) and  $n\mu_j \geq 0$  ( $j = r+1, \dots, t$ ) for all positive integers  $n$ . This implies  $\mu_i > 0$  for  $i = 1, \dots, r, \dots, t$ . Hence we have  $f(u) \leq f(e)$ ,  $f(u) \in \Delta$ . Since  $u$  is written as  $u = yz^{-1}$  for some  $y, z \in \Sigma$ , we have  $y = uz$ ,  $f(y) = f(u)f(z)$ . By using the condition  $(1^\circ)$  we can choose an element  $c \in C$  such that  $y = cz$ . Thus we obtain  $u = c \leq e$  as desired.

$(2) \Rightarrow (3)$ : Let  $a$  be an arbitrary element of  $S^*$ , and let  $b = az \in C$ ,  $z \in \Sigma$ . Then since  $e/b \geq e$ , we have  $b^* = e/(e/b) \leq e/e = e$ . Hence we obtain  $a \circ z = b^* \in C^* = \{c^*; c \in C\} = S^* \wedge C$ . Thus in order to prove that  $(S^*, \circ)$  is a group, it is sufficient to show that every element of  $C^*$  is invertible with respect to the operation " $\circ$ ". Let  $a \in C^*$ , and let  $u \in G$  be an element such that  $a(e/a) \leq u$ . Then  $u^{-1}a(e/a) \leq e$ ,  $u^{-1}a \leq e/(e/a) = a^* = a$ . Hence we have  $a \leq ua$ ,  $a \leq u^n a$  for  $n = 1, 2, \dots$ . If we take an element  $x \in \Sigma$  such that  $x \leq a$ , then  $x \leq u^n a$ . Hence  $u^{-n}x \leq a \leq e$  for  $n = 1, 2, \dots$ . This implies  $u^{-1} \leq e$ ,  $e \leq u$ . Thus we get  $e \leq \inf U(a(e/a)) = (a(e/a))^* \leq e^* = e$ . We obtain therefore  $a \circ (e/a) = e$ .

$(3) \Rightarrow (1_r)$ : Suppose that  $S$  is Artinian. Then  $(S^*, \circ, \vee, \wedge)$  is a cl-group and so  $(S^*, \circ, \vee, \wedge)$  is commutative l-group. For an element  $p^*$  of  $S^*$ ,  $p^*$  is irreducible if and only if  $p^*$  is prime. Since  $C$

satisfies the ascending chain condition in the sense of quasi-equality, each element of  $C^*$  is uniquely decomposed into primes apart from its commutativity. Now we show that  $(C, C^*, *)$  is a divisor theory of  $C$ . Suppose that  $x^* = a^* \circ y^*$  for  $x, y \in \Sigma$  and  $a^* \in C^*$ . Then since  $x^* = x$ ,  $y^* = y$  we have  $x = a^* \circ y$ ,  $xy^{-1} = x \circ y^{-1} = a^*$ ,  $x = a^* y$ . This shows that the condition (1°) holds for  $C$ . Let  $\Sigma(a^*)$  be the set of the elements  $x \in \Sigma$  which are divisible by  $a^*$ , i.e.,  $x \leq a^*$ . If  $\Sigma(a^*) = \Sigma(b^*)$  we obtain  $a^* = \sup \Sigma(a^*) = \sup \Sigma(b^*) = b^*$ . That is, the condition (2°) holds for  $C$ .

Similarly we can show the implications:  $(1_1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1_1)$ .

4. Uniqueness for divisor theory. Let  $(L, \Delta, f)$  be a (right) divisor theory of  $L$ . An element  $\alpha$  is called a principal divisor, if there is an element  $x \in \Sigma$  such that  $\alpha = f(x)$ . It is easily shown that  $\Sigma(\alpha)$  is not vacuous for each divisor  $\alpha$ .

UNIQUENESS THEOREM. For any two right divisor theories  $(L, \Delta, f)$  and  $(L, \Gamma, g)$  of  $L$  there exists an isomorphism  $\varphi$  from  $\Delta$  to  $\Gamma$ , under which the principal divisors in  $\Delta$  and in  $\Gamma$  correspond.

Proof. We shall show first that for each prime  $\pi \in \Delta$  there is a prime  $\rho \in \Gamma$  such that  $\Sigma(\rho) \subseteq \Sigma(\pi)$ . For, if not, there is a prime  $\pi \in \Delta$  for which there is no prime  $\rho \in \Gamma$  with  $\Sigma(\rho) \subseteq \Sigma(\pi)$ . Take an element  $x \in \Sigma(\pi)$ , and let  $g(x) = \rho_1^{k_1} \dots \rho_n^{k_n}$  be the prime factorization of  $g(x)$  in  $\Gamma$ . Then since each  $\Sigma(\rho_i)$  is not contained in  $\Sigma(\pi)$ , we can choose  $x_i$  which is contained in  $\Sigma(\rho_i)$  and not contained in  $\Sigma(\pi)$ . Hence there are  $\gamma_i \in \Gamma$  such that  $g(x_i) = \rho_i \gamma_i$  for  $i = 1, \dots, n$ . Then we have  $g(x_1^{k_1} \dots x_n^{k_n}) = g(x) \gamma$ , where  $\gamma = \gamma_1^{k_1} \dots \gamma_n^{k_n}$ . Hence by (1°)  $x_1^{k_1} \dots x_n^{k_n}$  is divisible by  $x$  on the right-hand side in  $L$ , hence  $f(x_1^{k_1} \dots x_n^{k_n})$  is divisible by  $f(x)$ , and hence  $f(x_1)^{k_1} \dots f(x_n)^{k_n}$  is divisible by  $\pi$ . Therefore  $\Sigma(\pi)$  contains some  $x_i$ , which is a contradiction. Symmetrically for the prime  $\rho \in \Gamma$ , there is a prime  $\pi' \in \Delta$  such that  $\Sigma(\pi') \subseteq \Sigma(\rho)$ .

Next we show that  $\pi = \pi'$ . Since  $\Delta$  is a semigroup with the

unique factorization theorem, we have  $\pi\pi' \neq \pi'$ . By using (2°) we can see that  $\sum(\pi\pi')$  is strictly contained in  $\sum(\pi')$  and hence in  $\sum(\pi)$ . Then we can take an element  $y \in \sum$  such that  $f(y)$  is divisible by  $\pi'$  and not divisible by  $\pi\pi'$ . If  $\pi \neq \pi'$ ,  $f(y)$  is divisible by  $\pi\pi'$ , since  $f(y)$  is divisible by  $\pi$ . This is impossible. We have therefore  $\pi = \pi'$ ,  $\sum(\pi) = \sum(\rho)$ . By using (2°) we can see easily that for each prime  $\pi \in \Delta$ , the prime  $\rho \in \Gamma$  with  $\sum(\rho) = \sum(\pi)$  is uniquely determined. Hence we can define the map  $\varphi : \pi \mapsto \rho = \varphi(\pi)$ . It is evident that  $\varphi$  extends uniquely to an isomorphism from  $\Delta$  to  $\Gamma$ .

In order to prove the last part of the theorem we suppose that  $f(x)$  is exactly divisible by  $\pi^k$ . Since  $\sum(\pi^2)$  is, by (2°), strictly contained in  $\sum(\pi)$ , we can choose an element  $x_0$  such that  $f(x_0) = \pi\alpha$  and  $\alpha$  is not divisible by  $\pi$ . Hence again by (2°) we can take an element  $u$  which is contained in  $\sum(\alpha^k)$  and not in  $\sum(\pi\alpha^k)$ . Then of course  $g(u)$  is not divisible by  $\varphi(\pi)$ . Since  $f(xu) = f(x)f(u) = \pi^k \alpha^k \beta = (\pi\alpha)^k \beta = f(x_0)^k \beta = f(x_0^k) \beta$  for some  $\beta \in \Delta$ , we get  $xu = bx_0^k$  for some  $b \in L$ . Hence we have  $g(x)g(u) = g(x_0^k)g(b)$ . On the other hand since  $g(x_0)$  is divisible by  $\varphi(\pi)$  and  $g(u)$  is not divisible by  $\varphi(\pi)$ ,  $g(x)$  is divisible by  $\varphi(\pi)^k$ . By a symmetrical argument we can show that  $f(x)$  is exactly divisible by  $\pi^k$  if and only if  $g(x)$  is exactly divisible by  $\varphi(\pi)^k$ . This completes the proof.

COROLLARY 1. Suppose that  $S$  is an Artinian 1-semigroup, and  $C$  the cone of  $S$ . Then the divisor theory  $(C, C^*, *)$  is uniquely determined apart from isomorphism.

COROLLARY 2.  $S$  and  $C$  are as same as in Corollary 1. Assume that the ascending chain condition holds for elements of  $C$ , and any prime element is maximal (in  $C$ ). Then  $S$  forms a commutative 1-group.

Proof. It can be proved that quasi-equality implies equality, which is similar to the proof of Theorem 2.6 in [1].



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